

# Global Mackey functors with operations and $n$ -special lambda rings

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# 1 Introduction

Mackey functors were introduced by Dress and Green [Dre73], [Dre71], [Gre71], following earlier ideas of Bredon and Lam [Bre67], [Lam68]. Excellent contemporary expositions are [Web00] and [Pan07]. Inspired by the very first steps of the theory, we take the point of view in [Bre67], that these theories formalize the algebraic properties of coefficient systems of equivariant (generalized) cohomology theories.

Many prominent examples of Mackey functors arise in this manner or in a closely related context. For instance, the representation rings form the coefficients of topological  $K$ -theory, group cohomology forms the coefficients of ordinary cohomology, and Burnside rings are closely related to cohomotopy. We limit our attention to finite groups.

Over the last decades, our understanding of generalized cohomology has evolved. Cohomology operations have emerged as an important part of the theory. There is, in particular, Atiyah’s work, putting the Adams operations in  $K$ -theory into a common framework with Steenrod’s operations in cohomology [Ati66], [Ada62], [Ste62]. Axioms for these ‘*power operations*’ were formulated by Bruner, May, McClure and Steinberger<sup>1</sup> in [BMMS86], and the modern foundations of homotopy theory are set up in such a way, that ‘nice enough’ multiplicative cohomology theories automatically have power operations. We would especially like to draw the reader’s attention to the recent work of Stefan Schwede [Sch]. We understand that Schwede’s *orthogonal spectra* define genuinely equivariant theories with power operations, whose coefficient systems form a large class of examples for the formalism formulated below.

A *global power functor* is a global Mackey functor with the extra structure of power operations. This definition captures the algebraic properties of the coefficient system of a (genuinely) equivariant theory with cohomology operations. We note that there is recent related work by Strickland on *Tambara functors* [Str12] and that Schwede has independently arrived at a notion of global power functor very similar to ours.

The paper is organized as follows: systematically using the language of groupoids, we start with an exposition of the theory of *global Mackey and Green functors*. These are discussed in [Web00], where Webb attributes the definition to Bouc [Bou96], referencing also older work of Symonds [Sym91]. What is different in our setup is the groupoid point of view. This is by no account a new idea, see for instance [Str00] and especially [Pan07]. Since we do, however, heavily use this formulation, we felt it worth to spell out the axioms and to give a full comparison to the definition of global Mackey functor found in [Web00].

We then proceed to define global power functors. Given such a global

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<sup>1</sup>Bruner, May, McClure and Steinberger work in the Borel-equivariant context, but the formulation for other equivariant theories is an easy translation exercise, see for instance [Gan06] or [Gan].

power functor, one can study rings with similar operations. For instance, special  $\lambda$ -rings are rings with operations similar to those on the representation rings  $R(G)$ . The precise concept is that of a  $\tau$ -ring, formulated by Hoffman, who proved that a  $\tau$  ring with respect to  $R$  is the same thing as a special  $\lambda$ -ring [Hof79]. Retelling Hoffman's story with  $R$  replaced by a different global power functor  $M$ , we arrive at a theory of rings with operations parametrized by elements of  $\bigoplus_{n \geq 0}^{\wedge} M(S_n)$  or, depending on some technical properties of  $M$ , by elements of  $\bigoplus_{n \geq 0}^{\wedge} M(S_n)^*$ . These  $\tau$ -rings with respect to  $M$  turn out to be non-special  $\lambda$ -rings with additional structure.

A key player in [Hof79] is the Schur-Weyl isomorphism

$$\bigoplus_{n \geq 0}^{\wedge} R(S_n) \xrightarrow{\cong} \lim_{m \in \mathbb{N}} R(U(m)), \quad (1)$$

defined and studied via  $\tau$ -operations, and we will discuss some attempts to generalize (1).

The example of the Burnside rings has been studied extensively, the resulting ' $\tau$ -rings' have become known as  $\beta$ -rings, see in particular [Rym77], but also [Boo75], [Och88], [MW84], [Val93], [Gui06]. Other examples include  $n$ -special  $\lambda$ -rings, involving class functions on  $n$ -tuples of commuting elements, 'subgroup class functions', where permutation representations have their characters, and an elliptic picture, where we suggest the construction of an elliptic analogue of the Schur-Weyl map (1), taking values in Looijenga's ring of symmetric theta functions [Loo77]. Rezk has independently developed a theory of '*elliptic  $\lambda$ -rings*', and we make contact with his work in 4.3.6.

A variation where  $R(S_n)$  is replaced with  $R^-(S_n)$ , the Grothendieck group of projective representations with Schur cocycle, gives rise to the notion of super  $\lambda$ -ring, fitting Sergeev-Yamaguchi duality into Hoffman's framework for (1).

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## 2 Globally defined Mackey functors

### 2.1 Finite groupoids

For a detailed introduction to groupoids see [Moe02] or [Str00]. A finite groupoid is a category  $\underline{G}$  with finitely many objects and finitely many morphisms and such that each morphism is invertible. We will write  $G_0$  for the set of objects of  $\underline{G}$  and  $G_1$  for the set of morphisms. To avoid confusion with maps between groupoids, we will refer to elements of  $G_1$  as *arrows*. Let  $g \in G_1$  be an arrow. We will write  $s(g)$  for the source and  $t(g)$  for the target of  $g$ . Given an object  $x \in G_0$ , we will refer to the automorphism group of  $x$  as the *stabilizer*  $\text{Stab}(x)$  of  $x$ . We write  $[\underline{G}]$  for the set of isomorphism classes of  $\underline{G}$ . If  $G$  is a finite group, we will write  $G$  also for the groupoid with  $G_0 = \text{pt}$  and  $G_1 = G$ . We choose this over the more common notation  $\text{pt} // G$ , since Mackey functors are traditionally applied to groups. So, for instance, the representation ring of  $G$  remains  $R(G)$ , rather than  $R(\text{pt} // G)$ .

A *map of groupoids*  $f: \underline{H} \rightarrow \underline{G}$  is a functor. We will write  $\mathcal{G}\text{pd}_{\text{fin}}$  for the category of finite groupoids and maps of groupoids. One can (and should) view  $\mathcal{G}\text{pd}_{\text{fin}}$  as 2-category with the natural isomorphisms as 2-morphisms, but in this paper we will not emphasize the 2-categorical point of view. A map  $f$  is an *equivalence* if it is an equivalence of categories, and  $f$  is called *faithful* if it is injective on stabilizers.

Fix a finite group  $G$ . Let  $G\text{Sets}_{\text{fin}}$  be the category of (left)  $G$ -sets. We write

$$\begin{aligned} G\text{Sets}_{\text{fin}} &\longrightarrow \mathcal{G}\text{pd}_{\text{fin}} \\ X &\longmapsto G \ltimes (-) \end{aligned}$$

for the functor that sends a  $G$ -set  $X$  to its translation groupoid. To fix notation, we say that  $G \ltimes X$  has objects  $X$  and an arrow

$$gx \leftarrow x$$

for each pair  $(g, x)$  in  $G \times X$ .

**Example 2.1.** The translation groupoid of the one point space is isomorphic to the group  $G$  itself. More generally, let  $H$  be a subgroup of  $G$ , and consider the left  $G$ -set  $G/H$ . The translation groupoid of  $G/H$  is canonically equivalent to the group  $H$ .

**Definition 2.2** (Inertia groupoids). Let  $\underline{G}$  be a groupoid. The *inertia groupoid*

$$\mathcal{I}(\underline{G}) = \text{hom}(\mathbb{Z}, \underline{G})$$

(sometimes also called *loop groupoid*) of  $\underline{G}$  has as objects the automorphisms of  $\underline{G}$ . An arrow in  $\mathcal{I}(\underline{G})$  from  $g \in \text{Stab}(x)$  to  $g' \in \text{Stab}(y)$  is an arrow  $h \in G_1$  from  $x$  to  $y$  satisfying  $g' = hgh^{-1}$ . Composition is defined in the obvious way.

**Definition 2.3** (Strict pullbacks). Consider a diagram of groupoids

$$\underline{K} \xrightarrow{\delta} \underline{G} \xleftarrow{\gamma} \underline{H}. \quad (2)$$

The pullback groupoid  $\underline{P}$  of (2) has object set

$$P_0 = K_0 \times_{G_0} H_0,$$

and similarly for arrows and the source, target and composition maps.

We have a commuting diagram

$$\begin{array}{ccc} \underline{P} & \xrightarrow{\alpha} & \underline{H} \\ \beta \downarrow & & \downarrow \gamma \\ \underline{K} & \xrightarrow{\delta} & \underline{G}, \end{array}$$

and  $\underline{P}$  is universal with respect to the property that it fits into this diagram.

**Definition 2.4.** A diagram that is isomorphic to one obtained in this fashion is called a *pullback diagram*.

**Definition 2.5** (Fibered products). The *homotopy pullback* or *fibered product*  $\underline{F}$  of (2) has objects

$$F_0 := \{(y, g, z) \mid y \in K_0, z \in H_0, g: \delta(y) \rightarrow \gamma(z)\}$$

and arrows

$$\begin{array}{ccccc}
 y & & \delta(y) & \xrightarrow{g} & \gamma(z) & & z \\
 \downarrow k & & \downarrow \delta(k) & & \downarrow \gamma(h) & & \downarrow h \\
 y' & & \delta(y') & \xrightarrow{g'} & \gamma(z') & & z'.
 \end{array}$$

Composition of arrows is defined in the obvious way.

The projections  $\alpha: \underline{F} \rightarrow \underline{H}$  and  $\beta: \underline{F} \rightarrow \underline{K}$  make the *homotopy pullback diagram*

$$\begin{array}{ccc}
 \underline{F} & \xrightarrow{\alpha} & \underline{H} \\
 \beta \downarrow & & \downarrow \gamma \\
 \underline{K} & \xrightarrow{\delta} & \underline{G},
 \end{array}$$

commute up to a natural transformation

$$\gamma \circ \alpha \xRightarrow{\cong} \delta \circ \beta,$$

and the triple  $(\underline{F}, \alpha, \beta)$  is universal with respect to that property.

**Definition 2.6.** We will refer to any diagram that is equivalent to a diagram obtained in this fashion as a *homotopy pullback diagram*.

Note that the pullback  $\underline{P}$  is contained in  $\underline{F}$  as the full subgroupoid whose objects are of the form  $(y, 1, z)$ . Fibered products are well-behaved under equivalences, whereas pullbacks are not.

**Example 2.7** (See [Pan07, Chap.0]). Assume that  $K$ ,  $G$ , and  $H$  are finite groups and that  $\delta$  and  $\gamma$  are injective maps of groups. Then the isomorphism classes of the fibered product  $\underline{F} = K \times_G H$  are in one to one correspondence with the double cosets  $H \backslash G / K$ . For  $g \in G$  the stabilizer of  $g$  in  $\underline{F}$  is isomorphic to

$$H \cap (gKg^{-1}).$$

Under this identification the map  $\alpha: \underline{F} \rightarrow H$  becomes the inclusion of  $H \cap (gKg^{-1})$  in  $H$ . Similarly,  $\beta$  becomes the inclusion of  $(g^{-1}Hg) \cap K$  in  $K$ , and the isomorphism that connects the source of  $\alpha$  with the source of  $\beta$  is conjugation by  $g$ .

**Remark 2.8.** The functor that sends (2) to its pullback square is the right Kan extension along the map of underlying diagrams. With an appropriate model structure on  $\mathcal{G}\text{pd}_{\text{fin}}$  [Str00, 6.12], the homotopy pullback becomes the right homotopy Kan extension along the same map of diagrams. In other words, forming the homotopy pullback (square) is the right derived functor of forming the pullback (square).

**Proposition 2.9.** *The functor  $G \ltimes (-)$  sends pullback diagrams in  $G\text{Sets}_{\text{fin}}$  to homotopy pullback diagrams in  $\mathcal{G}\text{pd}_{\text{fin}}$ .*

PROOF : In fact, the functor  $G \ltimes (-)$  preserves pullback squares. We claim that in this case the inclusion map  $\iota: \underline{P} \rightarrow \underline{F}$  is an equivalence of groupoids. We need to show that  $\iota$  is essentially surjective. Pick an object  $(y, g, z)$  of  $\underline{F}$ . Then the arrow

$$\begin{array}{ccccc} gy & & \delta(gy) & \xrightarrow{1} & \gamma(z) & & z \\ g^{-1} \downarrow & & g^{-1} \downarrow & & \downarrow 1 & & \downarrow 1 \\ y & & \delta(y) & \xrightarrow{g} & \gamma(z) & & z \end{array}$$

gives the desired isomorphism from an object in the image of  $\iota$ .  $\square$

## 2.2 Mackey functors

Let  $\mathcal{A}$  be an additive category. Let  $M$  be a pair  $M = (M_*, M^*)$  of functors

$$\mathcal{G}\text{pd}_{\text{fin}} \longrightarrow \mathcal{A},$$

agreeing on objects, where  $M_*$  is covariant and  $M^*$  is contravariant. We write

$$M(\underline{G}) := M_*(\underline{G}) = M^*(\underline{G})$$

for the effect of  $M$  on objects and

$$\phi_* := M_*(\phi) \quad \text{and} \quad \phi^* := M^*(\phi)$$

for its effect on morphisms. We call  $M$  a *globally defined Mackey functor* if the following axioms are satisfied:

**Coproducts:**  $M_*$  preserves coproducts, and  $M(\emptyset) = 0$ .



**Natural isomorphisms:** If  $\phi$  and  $\psi$  are two naturally isomorphic maps of groupoids  $\underline{G} \rightarrow \underline{H}$ , then we have  $\phi_* = \psi_*$  and  $\phi^* = \psi^*$ .

**Equivalences:** If  $\phi: \underline{G} \xrightarrow{\sim} \underline{H}$  is an equivalence of groupoids, then  $\phi_*$  and  $\phi^*$  are inverse isomorphisms.

**Fibered Products:** For any diagram of groupoids

$$\begin{array}{ccc} \underline{F} & \xrightarrow{\alpha} & \underline{H} \\ \beta \downarrow & & \downarrow \gamma \\ \underline{K} & \xrightarrow{\delta} & \underline{G}, \end{array}$$

where  $\underline{F}$  is the *fibered product* of  $\underline{H}$  and  $\underline{K}$  over  $\underline{G}$ , we have the *push-pull identity*

$$\delta^* \circ \gamma_* = \beta_* \circ \alpha^*.$$

A *natural transformation*  $\eta: M \rightarrow N$  of Mackey functors is a family of maps

$$\eta_{\underline{G}}: M(\underline{G}) \longrightarrow N(\underline{G}) \quad \underline{G} \in \text{ob}(\text{Gpd})$$

that is natural with respect to both, the co- and the contravariant structure of  $M$  and  $N$ .

The following lemma is an immediate consequence of Proposition 2.9.

**Lemma 2.10.** *Let  $M$  be a globally defined Mackey functor with values in  $\mathcal{A}$ . Then the composite*

$$M \circ G \ltimes (-) : G\text{Sets}_{\text{fin}} \longrightarrow \mathcal{A}$$

*is a Mackey functor for  $G$  in the sense of Dress.*

In the appendix, we prove that our definition becomes equivalent to Webb's definition of globally defined Mackey functor [Web00, 8] if one adds one more axiom to the above list:

**Surjections axiom:** Let  $\phi: G \rightarrow H$  be a surjective map of groups. Then

$$\phi_* \phi^* = \text{id}_{M(H)}.$$

## 2.3 Consequences of the axioms

1. Let  $G$  be a finite group and let  $H$  be a subgroup of  $G$ . Then the canonical equivalence of groupoids

$$H \xrightarrow{\sim} \underline{G/H},$$

induces an isomorphism

$$M(H) \xrightarrow{\cong} M(\underline{G/H}).$$

2. Let  $G$  be a finite group and let  $c^g: G \rightarrow G$  be an inner automorphism of  $G$ . Then  $c^g$  is naturally isomorphic to  $\text{id}_G$ , and hence

$$(c^g)_* = \text{id}_{M(G)} = (c^g)^*.$$

3. Let  $\alpha: \underline{G} \xrightarrow{\sim} \underline{H}$  be an equivalence of groupoids with quasi-inverse  $\beta: \underline{H} \xrightarrow{\sim} \underline{G}$ . Then the composite  $\alpha \circ \beta$  is naturally isomorphic to  $\text{id}_{\underline{H}}$ , and we get

$$\alpha_* = (\beta_*)^{-1} = \beta^*.$$

4. Let  $\underline{G} = \underline{H} \sqcup \underline{K}$ , and let  $\underline{\alpha}: \underline{H} \rightarrow \underline{G}$  and  $\underline{\beta}: \underline{K} \rightarrow \underline{G}$  be the inclusions. Then  $\alpha^*$  and  $\beta^*$  are the projections onto the summands of

$$M(\underline{G}) \cong M(\underline{H}) \oplus M(\underline{K}),$$

while  $\alpha_*$  and  $\beta_*$  are the inclusions of the summands.

PROOF OF (4): The fibred product  $\underline{K} \times_{\underline{G}} \underline{H}$  is empty, so we get

$$\beta^* \alpha_* = 0 \quad \text{and} \quad \alpha^* \beta_* = 0.$$

Further, we have  $\underline{H} \times_{\underline{G}} \underline{H} \cong \underline{H}$  and similarly for  $\underline{K}$ , implying

$$\alpha^* \alpha_* = \text{id}_{M(\underline{H})} \quad \text{and} \quad \beta^* \beta_* = \text{id}_{M(\underline{K})}.$$

□

### 2.3.1 Variations

In [Web00], Webb describes various variations of the definition of global Mackey functor. His definition depends on two classes of finite groups, which he denotes  $\mathcal{X}$  and  $\mathcal{Y}$ . Above we have only formulated the strongest case, where both classes contain all finite groups. One can adapt our definition to other cases by imposing the appropriate conditions on stabilizers. For instance, for the case where  $\mathcal{Y}$  only contains the trivial group, we can weaken our definition by requiring  $M_*(\alpha)$  to be defined only for faithful  $\alpha$ .

## 2.4 Examples

### 2.4.1 Burnside rings

Let  $Sets^f$  denote the category of finite sets. Then the category of  $\underline{G}$ -sets (or ‘sheaves in finite sets’ over  $\underline{G}$ ) is the functor category

$$Sh(\underline{G}) := Fun(\underline{G}, Sets^f).$$

Let  $\phi: \underline{G} \rightarrow \underline{H}$  be a map of finite groupoids. Then we have an adjoint functor pair

$$\phi_* : Sh(\underline{G}) \rightleftarrows Sh(\underline{H}) : \phi^{-1},$$

with

$$\phi^{-1}\mathcal{F} = \mathcal{F} \circ \phi \quad \text{and} \quad \phi_*\mathcal{F} = R\text{Kan}_\phi \mathcal{F}$$

(left Kan-extension, compare [Moe02, 6.3]). The Burnside ring of  $\underline{G}$  is the Grothendieck group

$$A(\underline{G}) = (Sh(\underline{G}) / \cong, \sqcup)^{gp}.$$

The discussion in [Pan07] implies that  $A$  is a global Mackey functor satisfying the surjectivity axiom.

**Example 2.11.** If  $G$  is a finite group, then  $Sh(G)$  is naturally identified with the category of finite left- $G$ -sets, and  $A(G)$  is the Burnside ring of  $G$ .

### 2.4.2 Group cohomology

The Borel construction functor (c.f. [Moe02])

$$\text{Borel}: \mathcal{G}\text{pd}_{\text{fin}} \longrightarrow \mathcal{S}$$

from the category of (finite) groupoids into the stable homotopy category has transfers along faithful maps of groupoids, making

$$\underline{G} \longmapsto H^*(B\underline{G}; \mathbb{Z})$$

a global Mackey functor, where the  $\phi_*$  are only defined for faithful  $\phi$ . The degree zero part

$$H^0(B\underline{G}; \mathbb{Z}) \cong \mathbb{Z}^{[\underline{G}]}$$

counts the number of isomorphism classes of  $\underline{G}$ .

**Lemma 2.12.** *Let  $f$  be a  $\mathbb{Z}$ -valued function on  $[\underline{G}]$ , and let  $\phi: \underline{G} \rightarrow \underline{H}$  be a faithful map. Then*

$$\phi_* f: [\underline{H}] \longrightarrow \mathbb{Z}$$

*satisfies*

$$\frac{(\phi_* f)(y)}{|\text{aut}_{\underline{H}}(y)|} = \sum_{[x] \mapsto [y]} \frac{f(x)}{|\text{aut}_{\underline{G}}(x)|}. \quad (3)$$

*Proof.* This follows from the homotopy pullback square

$$\begin{array}{ccc} \coprod_{[x] \mapsto [y]} \text{aut}(y)/\text{aut}(x) & \longrightarrow & \underline{G} \\ \downarrow & & \downarrow \\ \text{pt} & \xrightarrow{y} & \underline{H}. \end{array}$$

□

We may replace  $\mathbb{Z}$  with any other abelian ring  $R$ . If  $R$  is a  $\mathbb{Q}$ -algebra then  $H^0(-; R)$  possesses transfers along all maps, defined by (3).

**Example 2.13.** Let  $G$  be a finite group. Then  $BG$  is a classifying space of  $G$ , and  $H^*(BG; \mathbb{Z})$  is group cohomology of  $G$  with trivial coefficients.

**Example 2.14.** More generally, let  $X$  be a finite  $G$ -set. Then

$$B(G \ltimes X) \simeq EG \times_G X,$$

and  $H^*(EG \times_G X; \mathbb{Z})$  is the Borel equivariant cohomology of  $X$ .

### 2.4.3 Representation rings

We write  $\text{Vect}_{\mathbb{C}}^f$  for the category of finite dimensional complex vector spaces and consider the functor category

$$\text{Rep}_{\mathbb{C}}(\underline{G}) = \mathcal{F}un(\underline{G}, \text{Vect}_{\mathbb{C}}^f)$$

of complex  $\underline{G}$  representations. Let  $\phi: \underline{G} \rightarrow \underline{H}$  be a map of groupoids. Then there is an adjoint functor pair

$$\text{ind}_{\phi}: \text{Rep}_{\mathbb{C}}(\underline{G}) \rightleftarrows \text{Rep}_{\mathbb{C}}(\underline{H}) : \text{res}_{\phi},$$

where  $\text{res}_{\phi}$  is precomposition with  $\phi$ , while  $\text{ind}_{\phi}$  is left Kan extension along  $\phi$ . We write  $R(\underline{G})$  for the Grothendieck group of the category of representations of  $\underline{G}$ . The maps induced by  $\text{ind}$  and  $\text{res}$  make  $R$  into a global Mackey functor satisfying the surjectivity axiom<sup>2</sup>. For a fixed groupoid  $\underline{G}$ , we also have the  $\underline{G}$ -Mackey functor sending  $\underline{H}$  to the  $H^2(B\underline{G}; U(1))$ -graded ring of projective representations,

$$R^*(\underline{H}) = \bigoplus_{\alpha} R^{\phi^*(\alpha)}(\underline{H}).$$

**Example 2.15.** If  $\phi$  is an inclusion of finite groups, then  $\text{ind}_{\alpha}$  is the familiar induced representation functor.

**Example 2.16.** If  $\phi$  is the unique map from a finite group  $G$  to the trivial group, then  $\text{ind}_{\phi}$  is the inner product with one, i.e.,  $\text{ind}_{\phi} \rho$  is the trivial summand of  $\rho$ .

**Example 2.17.** Let  $G$  be a finite group acting on a finite set  $X$ , and let  $\underline{G} = G \ltimes X$  be the corresponding translation groupoid. Then the category of  $\underline{G}$ -representations is naturally identified with the category of  $G$ -equivariant vector bundles on  $X$ . Let  $f: X \rightarrow Y$  be a map of  $G$ -sets, and let  $\phi := G \ltimes f$ , then  $\text{ind}_{\phi}$  is the equivariant Atiyah transfer along  $f$ .

### 2.4.4 $K(n)$ -local cohomology theories

In [Str00], Strickland showed that the object  $B\underline{G}$  becomes self-dual in the  $K(n)$ -local category  $\mathcal{S}_{K(n)}$ , a localization of the stable homotopy category

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<sup>2</sup>A good reference for the groupoid formulation of this fact is [Pan07, Prop. 0.0.1].

$\mathcal{S}$ . Let  $E^*$  be a generalized cohomology theory that is well-defined on  $\mathcal{S}_{K(n)}$ , for instance, Morava-Lubin-Tate theory  $E_n^*$ . Then Strickland's work implies that

$$\underline{G} \longmapsto E^*(B\underline{G})$$

is a global Mackey functor with transfers along all maps. The surjections axiom does not hold in this example.

#### 2.4.5 n-Class functions

If  $M$  is a Mackey functor then so is  $M \circ \mathcal{I}$ , where  $\mathcal{I}$  denotes the inertia groupoid. It follows that, for any  $n \in \mathbb{N}$ , we have a Mackey functor

$$\text{n-Class}(\underline{G}, R) := H^0(B\mathcal{I}^n \underline{G}; R)$$

with transfers along faithful maps (transfers along all maps if  $R$  is a  $\mathbb{Q}$ -algebra). An element  $\chi \in \text{n-Class}(\underline{G}, R)$  is called an *n-class function* on  $\underline{G}$ . It is defined on  $n$ -tuples of commuting automorphisms of  $\underline{G}$  and satisfies

$$\chi(sg_1s^{-1}, \dots, sg_ns^{-1}) = \chi(g_1, \dots, g_n)$$

(invariance under simultaneous conjugation). For  $n \geq 2$  this does not satisfy the surjectivity axiom. The group  $\text{Aut}(\mathbb{Z}^n)$  acts on  $\mathcal{I}^n \underline{G} = \text{Hom}(\mathbb{Z}^n, \underline{G})$ , inducing an action by Mackey-functor automorphisms on  $\text{n-Class}(-, R)$ . We will also be interested in the Mackey functor

$$\underline{G} \longmapsto \text{n-Class}(\underline{G}; R)^{\text{Aut}(\mathbb{Z}^n)}.$$

**Definition 2.18.** For a topological space  $X$ , we write  $\text{Prin}_{\underline{G}}(X)$  for the category of principal  $\underline{G}$ -bundles over  $X$  and their isomorphisms over  $X$ .

If  $X$  is the  $n$ -torus

$$T^n = \mathbb{S}^1 \times \dots \times \mathbb{S}^1$$

then we have an  $\text{Aut}(\mathbb{Z}^n)$ -equivariant equivalence of groupoids

$$\mathcal{I}^n(\underline{G}) \simeq \text{Prin}_{\underline{G}}(T^n), \tag{4}$$

where the  $\text{Aut}(\mathbb{Z}^n)$ -action on the left-hand side is by automorphisms of  $T^n = B\mathbb{Z}^n$ . So,  $\text{n-Class}, R(\underline{G})$  may be interpreted as a ring of global functions on

$Prin_{\underline{G}}(T^n)$ , and  $n\text{-Class}(\underline{G})^{Aut(\mathbb{Z}^n)}$  as ring of  $Aut(\mathbb{Z}^n)$ -invariant such global functions.

A variation of these are the Mackey functors

$$\begin{aligned} \underline{G} &\longmapsto n\text{-Class}_p(\underline{G}, R) \\ \text{and } \underline{G} &\longmapsto n\text{-Class}_p(\underline{G}, R)^{Aut((\widehat{\mathbb{Z}}_p)^n)}, \end{aligned}$$

of *p-adic n-class functions*, defined exactly like  $n\text{-Class}$ , but with the inertia groupoid  $\mathcal{I}$  replaced by the *p-adic inertia groupoid*

$$I_p(-) = \mathcal{F}un(\widehat{\mathbb{Z}}_p, -).$$

So, elements of  $n\text{-Class}_p(\underline{G}, R)$  are functions defined on commuting *n*-tuples of *p*-power order automorphisms and invariant under simultaneous conjugation.

#### 2.4.6 Subgroup class functions

**Definition 2.19.** Let  $\underline{G}$  be a finite groupoid. Then the *subgroup groupoid* of  $\underline{G}$ , denoted  $\mathcal{S}(\underline{G})$  has as objects the subgroups of  $\underline{G}$  and arrows

$$H \xrightarrow{g} gHg^{-1}.$$

A *subgroup class function* (a.k.a. “*supercentral function*” in [Knu73]) with values in  $R$  is a function from  $\mathcal{S}(\underline{G})$  to  $R$ . The ring of all subgroup class functions on  $\underline{G}$  is denoted

$$SCF(\underline{G}, R) = R^{\underline{G}}.$$

If  $R$  is clear from the context, we may simply write  $SCF(\underline{G})$ .

**Remark 2.20.** The term “supercentral function” dates back to the time before “super” meant  $\mathbb{Z}/2\mathbb{Z}$ -graded.

Similarly to  $n\text{-Class}$ , one sees that  $SCF$  is a Mackey functor with transfers along faithful maps. If  $R$  contains  $\mathbb{Q}$ , then  $SFC$  has all transfers, but no surjections axiom.

### 2.4.7 Character maps

**Example 2.21.** We have a natural transformation of Mackey functors

$$\text{char}: R(\underline{G}) \longrightarrow 1\text{-Class}(\underline{G}; \mathbb{Q}[\mu_\infty])$$

defining an isomorphism onto its image

$$\text{char}: R(\underline{G}) \xrightarrow{\cong} 1\text{-Class}(\underline{G}; \mathbb{Z}[\mu_\infty])^{\widehat{\mathbb{Z}}}.$$

**Example 2.22.** Let  $E_n$  be the degree zero part of Morava-Lubin-Tate theory. Then we have a natural transformation of Mackey functors (with all transfers)

$$\text{char}: E_n(B\underline{G}) \longrightarrow n\text{-Class}_p(\underline{G}; L)$$

defining an isomorphism onto its image

$$\text{char}: E_n(B\underline{G}) \xrightarrow{\cong} n\text{-Class}_p(\underline{G}; L)^{\text{Aut}(\widehat{\mathbb{Z}}_p^n)},$$

see [HKR00, Thms C& D] and [Gan06, Prop.1.6].

Recall that the Burnside ring  $A$  is initial among the global Mackey functors with transfers along faithful maps: given a global Mackey functor  $M$ , the unique natural transformation of Mackey functors  $\eta: A \Rightarrow M$  is given by

$$\begin{aligned} \eta_G: A(G) &\longrightarrow M(G) \\ [G/H] &\longrightarrow \text{ind}_H^G(1). \end{aligned}$$

If  $M$  has all transfers *and the surjections axiom holds* then  $\eta$  preserves all transfers. We will see that if  $M$  is a global Green functor (resp. global power functor), defined below, then  $\eta$  is a transformation of Green functors (resp. power functors).

The following two examples should be compared to [Knu73, Chapter 2.4] and [Sol67].

**Example 2.23** (Subgroup character of a permutation representation). The transformation

$$\eta^{SC}: A \longrightarrow \text{SCF}(-, \mathbb{Z})$$

sends the  $G$ -set  $Y$  to the subgroup class function

$$\chi_Y: [H] \longmapsto |Y^H|$$

( $H$ -fixed points). Knutson proves that  $\eta$  is injective. The subgroup characters of the transitive  $G$ -sets are Burnside's '*marks*'.



**Example 2.24** (Chromatic character maps). The transformation

$$\eta^n: A \longrightarrow \text{n-Class}(-, \mathbb{Z})$$

sends the  $G$ -set  $X$  to the  $n$ -class function

$$\chi_X: (g_1, \dots, g_n) \longmapsto |X^{(g_1, \dots, g_n)}|.$$

The two last examples are related by the maps

$$\begin{aligned} \mathcal{I}^n(\underline{G}) &\longrightarrow \mathcal{S}(\underline{G}) \\ (g_1, \dots, g_n) &\longmapsto \langle g_1, \dots, g_n \rangle, \end{aligned}$$

inducing the transformations of Mackey functors

$$SCF(\underline{G}, \mathbb{Z}) \longrightarrow \text{n-class}(\underline{G}, \mathbb{Z})^{Aut(\widehat{\mathbb{Z}}^n)}.$$

For  $n = 1$ , we have the commuting diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta} & 1\text{-Class}(-, \mathbb{Z}[\mu_\infty]) \\ & \searrow \eta & \nearrow char \\ & R & \end{array}$$

and  $\chi_X$  is the character of the permutation representation  $G \rightarrow S_{|X|}$ . Similarly, for  $n = 2$ , we have the permutation 2-representation defined by  $X$  with the trivial 2-cocycle, and its 2-character is  $\chi_X$ , see [GK08]. In the groupoid picture,  $\chi_X$  counts the size of the fibers of the map

$$[\mathcal{I}^n(\underline{G} \ltimes X)] \longrightarrow [\mathcal{I}^n(\underline{G})].$$

#### 2.4.8 $G$ -modular functions

Let  $\mathfrak{H}$  be the upper half plane and write

$$\mathcal{M}_{\underline{G}} = \mathcal{I}^2 \underline{G} \times_{SL_2 \mathbb{Z}} \mathfrak{H}$$

for the orbifold groupoid

$$SL_2 \mathbb{Z} \ltimes (\mathcal{I}^2 \underline{G} \times \mathfrak{H}).$$

The space  $\mathcal{M}_{\underline{G}}$  parametrizes principal  $\underline{G}$ -bundles over complex elliptic curves (see [Gan09], [Mor09], [Car10] . Global functions on  $\mathcal{M}_{\underline{G}}$  are functions  $F(g, h; \tau)$  satisfying

$$F(g^a h^c, g^b h^d; \tau) = F\left(g, h; \frac{a\tau + b}{c\tau + d}\right) \quad (5)$$

and

$$F(sgs^{-1}, shs^{-1}; \tau) = F(g, h; \tau). \quad (6)$$

For  $\alpha \in H^3(B\underline{G}; U(1))$ , we have the Freed-Quinn line bundle  $\mathcal{L}^\alpha$  over  $\mathcal{M}_{\underline{G}}$ , see [FQ93]. Sections of  $mL^\alpha$  may be interpreted as functions  $F(g, h; \tau)$  as above satisfying (5) up to a root of unity and satisfying (6), see [Gan09]. We write

$$Mod(\underline{G}) = \Gamma(\mathcal{M}_{\underline{G}}, \mathcal{O}_{\mathcal{M}_{\underline{G}}})$$

for the Mackey functor of  $\underline{G}$ -*modular functions* as in (5) and (6). Its transfer along  $\phi$  is given by the formula (3) applied to  $\mathcal{I}^2\phi$ . For fixed  $\underline{G}$ , we also define the  $\underline{G}$ -Mackey functor sending  $\phi: \underline{H} \rightarrow \underline{G}$  to the  $H^3(B\underline{G}; U(1))$ -graded ring

$$Mod^*(\underline{H}) = \bigoplus_{\alpha} \Gamma(\mathcal{M}_{\underline{H}}, \mathcal{L}^{\phi^*(\alpha)}).$$

A level 2 version of the Mackey functor  $Mod(\underline{G})$  was studied in [Dev96].

#### 2.4.9 Tate $K$ -groups

Let  $\xi$  be the natural automorphism of  $id_{\mathcal{I}\underline{G}}$  with

$$\xi_{(x,g)} = g.$$

The Mackey functor

$$K_{Tate}(\underline{G}) \subset R(\mathcal{I}\underline{G})$$

considered in [Gan] is the Grothendieck group of formal power series  $\sum V_n q^{\frac{n}{|\xi|}}$  such that  $\xi$  acts on  $V_n$  with eigenvalue  $e^{2\pi i n / |\xi|}$ . For fixed  $\underline{G}$ , the transgression map

$$\tau: H^3(B\underline{G}; U(1)) \longrightarrow H^2(B\mathcal{I}\underline{G}; U(1)),$$

defined, for instance, in [Wil08], gives rise to the twisted Tate  $K$ -groups

$$K_{Tate}^\alpha(\underline{G}) \subset R^{\tau(\alpha)}(\mathcal{I}\underline{G})[[q^{\frac{1}{h|\xi|}}]]$$

as follows: the cocycle  $\tau(\alpha)$  classifies an extension  $\widetilde{\mathcal{I}\underline{G}}$  of  $\mathcal{I}\underline{G}$ . A lift  $\widetilde{\xi}$  of  $\xi$  in  $\widetilde{\mathcal{I}\underline{G}}$  has order  $h|\xi|$  for some natural number  $h \geq 1$ . Then  $K_{Tate}^\alpha$  is the Grothendieck group of formal power series  $\sum V_n q^{\frac{1}{h|\xi|}}$  such that

$$V_n \in \text{Rep}_{\mathbb{C}}(\widetilde{\mathcal{I}\underline{G}})$$

is an eigenspace of  $\widetilde{\xi}$  with eigenvalue  $e^{2\pi i n/h|\xi|}$ . This gives rise to an  $H^3(B\underline{G}; U(1))$ -graded  $\underline{G}$ -Mackey functor  $K_{Tate}^*$ .

By the *q-expansion principle* of [FQ93, Sec.5] (see also [Gan09, 2.4]), we have a natural transformation of Mackey functors

$$\text{Mod}(-) \implies K_{Tate}(-) \otimes \mathbb{C}$$

and of  $H^3(B\underline{G}; U(1))$ -graded  $\underline{G}$ -Mackey functors

$$\text{Mod}^*(-) \implies K_{Tate}^*(-) \otimes \mathbb{C}.$$

Alternatively, we could interpret  $q$ -expansion as a map

$$\text{Mod}^\alpha(\underline{G}) \longrightarrow \bigoplus_{[g]} 1\text{-Class} \left( \text{aut}_{\widetilde{\mathcal{I}\underline{G}}}(g), \mathbb{Z}[\mu_\infty] \right) [[q^{\frac{1}{h|g|}}]]$$

and note that if the modular section  $F$  is invariant under the  $\text{Aut}(\widehat{\mathbb{Z}})$  action on each summand of the target, then the  $q$ -expansion actually takes values in  $K_{Tate}^\alpha(\underline{G})$ . This is the second condition of Norton's generalized Moonshine conjecture [Nor87].

## 3 Products

### 3.1 Green functors

Assume that the additive category  $\mathcal{A}$  comes equipped with a bilinear symmetric monoidal structure  $\otimes$  with unit  $A$ .

**Definition 3.1.** A *globally defined Green functor* with values in  $(\mathcal{A}, \otimes, A)$  is a globally defined Mackey functor  $M$ , satisfying  $M(\text{pt}) = A$  and such that the contravariant part,  $M^*$ , takes values in the category of commutative ring objects in  $\mathcal{A}$ .

**Proposition 3.2.** A *globally defined Green functor with values in  $(\mathcal{A}, \otimes, A)$  is the same thing as a globally defined Mackey functor  $M$  together with an isomorphism  $M(\text{pt}) \cong A$  and with unitary, commutative and associative Künneth maps*

$$\kappa: M(\underline{G}) \otimes M(\underline{H}) \longrightarrow M(\underline{G} \times \underline{H}), \quad (7)$$

making  $M^*$  a (lax) bimonoidal functor.

PROOF : Let  $M$  be a global Mackey functor. Then the Künneth map (7) is defined as the composite of

$$M(\underline{G}) \otimes M(\underline{H}) \xrightarrow{pr_{\underline{G}}^* \otimes pr_{\underline{H}}^*} M(\underline{G} \times \underline{H}) \otimes M(\underline{G} \times \underline{H})$$

with the multiplication of the ring object  $M(\underline{G} \times \underline{H})$ . Conversely, given Künneth maps, the multiplication on  $M(\underline{G})$  is given by

$$\delta^* \circ \kappa: M(-) \otimes M(-) \longrightarrow M(-)$$

and the unit by

$$\varepsilon^*: A \longrightarrow M(-).$$

Here  $\delta$  and  $\varepsilon$  are the natural transformations on  $Gpd$  sending  $\underline{G}$  to its diagonal map

$$\delta_{\underline{G}}: \underline{G} \longrightarrow \underline{G} \times \underline{G}$$

and to the unique map

$$\varepsilon_{\underline{G}}: \underline{G} \longrightarrow \text{pt}.$$

□

In almost all of our examples, the Künneth map  $\kappa$  turns out to be an isomorphism. The exception is the Burnside ring example, and in the  $K_{Tate}$ -case, we need to invert  $q$  to have a Künneth isomorphism.

### 3.1.1 Consequences of the definition

In all our examples of Green functors, it makes sense to speak of elements of objects of  $\mathcal{A}$ , so, to simplify notation, we will denote the product of two elements by  $s \cdot t$ . Let  $M$  be a global Green functor. Then the following hold.

**Frobenius axiom:** Let  $\phi: \underline{G} \rightarrow \underline{H}$  be a faithful map of groupoids. For elements  $s \in M(\underline{G})$  and  $t \in M(\underline{H})$ , we have

$$\phi_*(s) \cdot t = \phi_*(s \cdot \phi^*t)$$

and

$$t \cdot \phi_*s = \phi_*((\phi^*t) \cdot s).$$

*Proof.* The second equation follows from naturality of  $\kappa$  and the homotopy pullback square

$$\begin{array}{ccc} \underline{G} & \xrightarrow{\phi} & \underline{H} \\ (\phi \times \text{id}_{\underline{G}}) \circ \delta_{\underline{G}} \downarrow & & \downarrow \delta_{\underline{H}} \\ \underline{H} \times \underline{G} & \xrightarrow{\text{id}_{\underline{H}} \times \phi} & \underline{H} \times \underline{H}. \end{array}$$

The first equation is proved analogously. □

**Inner products:** We have a bilinear form

$$\langle r, s \rangle_{\underline{G}} = \varepsilon_{\underline{G},*}(r \cdot s)$$

on  $M(\underline{G})$ .

**Frobenius reciprocity:** Let  $\phi: \underline{G} \rightarrow \underline{H}$  be a faithful map. Then we have

$$\langle r, \phi_*s \rangle_{\underline{H}} = \langle \phi^*r, s \rangle_{\underline{G}}$$

### 3.2 The graded rings $S_M(G)$

**Definition 3.3.** The  $n$ th symmetric power of a groupoid  $\underline{G}$  is the groupoid  $S_n \int \underline{G}$  obtained from  $\underline{G}^n$  by adding arrows

$$(x_1, \dots, x_n) \xrightarrow{\sigma} (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$$

for  $\sigma \in S_n$  and  $(x_1, \dots, x_n) \in G_0^n$ , with the relation

$$\sigma(g_1, \dots, g_n) = (g_{\sigma^{-1}(1)}, \dots, g_{\sigma^{-1}(n)})\sigma.$$

This construction is functorial in  $\underline{G}$  and preserves equivalences. If  $\underline{G} = G$  is a group then  $S_n \int G$  is the wreath product of  $S_n$  with  $G$ . We have

$$S_0 \int \underline{G} = 1$$

and equivalences

$$\prod_{k=0}^n (S_k \int \underline{G}) \times (S_{n-k} \int \underline{H}) \xrightarrow{\sim} S_n(\underline{G} \sqcup \underline{H}). \quad (8)$$

Let  $M$  be a global Green functor, and let  $\underline{G}$  be a finite groupoid.

**Definition 3.4.** We write

$$S_M(\underline{G}) := \bigoplus_{n \geq 0}^{\wedge} M(S_n \int \underline{G}) t^n,$$

for the infinite product of the  $M(S_n \int \underline{G})$ . Here  $t$  is a dummy variable, keeping track of the grading. We will also write  $S_M$  for  $S_M(\text{pt})$ .

$S_M$  is bivariantly functorial, and (7) and (8) give a natural transformation

$$\kappa: S_M(\underline{G}) \otimes S_M(\underline{H}) \longrightarrow S_M(\underline{G} \sqcup \underline{H}).$$

There are two ring structures on  $S_M(\underline{G})$ , namely

**Juxtaproduct:** This is defined by degree-wise multiplication, using the ring structure of  $M(S_n \int \underline{G})$ . The juxtaproduct of two elements of different degrees is zero. The unit of the juxtaproduct is  $\mathbf{1} = \sum 1_n t^n$ , where  $1_n$  stands for  $1 \in M(S_n \int \underline{G})$ .

**Cross product:** The cross product is

$$\times: S_M(\underline{G}) \otimes S_M(\underline{G}) \xrightarrow{\kappa} S_M(\underline{G} \sqcup \underline{G}) \xrightarrow{S_{M*}(f)} S_M(\underline{G}),$$

where  $f: \underline{G} \sqcup \underline{G} \rightarrow \underline{G}$  is the fold map. This makes  $S_M(\underline{G})$  into a graded ring with unit  $1 = 1_0$ .

We will see below that  $(S_M, \times, 1_0)$  may be viewed as a ring of internal operations on  $M(\underline{G})$ , if  $M$  is a global power functor with all transfers.

The definition of  $S_M$  still makes sense if  $M$  is replaced by an arbitrary bivariant functor out of  $Gpd$ . So, we may iterate  $S$ , and find that

$$M \longmapsto S_M$$

is a comonad, whose comultiplication are

$$\nabla^*: S_M \Longrightarrow S_{S_M}$$

and counit

$$v: S_M \Longrightarrow M$$

where  $\nabla$  is made up of the canonical inclusions

$$S_n \int S_m \int \underline{G} \hookrightarrow S_{nm} \int \underline{G}$$

and  $v$  is projection onto the coefficient of  $t^1$ . If

$$\kappa: M(S_n \int \underline{G}) \otimes M(S_m \int \underline{G}) \longrightarrow M((S_n \int \underline{G}) \times (S_m \int \underline{G}))$$

is an isomorphism for all  $m, n$ , then  $S_M(\underline{G})$  is a Hopf algebra over  $M(\text{pt})$  whose coproduct is the pullback along (8) (see [Hof79, Thm 1.2]).

### 3.2.1 The elements $f_n$ and $c_n$

Let  $M$  be a global Green functor,  $\mathbf{1} = \sum 1_n t^n$  the unit of the juxtaproduct on  $S_M$ . There are two important sequences of elements of  $S_M$ . Let

$$f(t) = \sum_{n \geq 0} f_n t^n$$

be the power series with

$$\mathbf{1} \times f(-t) = \mathbf{1}.$$

This determines the  $f_n \in M(S_n)$  inductively. The elements  $c_k \in M(S_k)$  for  $k \geq 1$  are defined by

$$\sum_{k \geq 1} c_k t^{k-1} = \frac{d}{dt} \ln \mathbf{1},$$

where the logarithmic power series  $\ln$  is computed using the cross product.

### 3.3 Examples

#### 3.3.1 Burnside rings

The Burnside ring functor is a global Green functor with the Künneth map given by the Cartan product of sets. The ring

$$\mathbf{B} = \bigoplus_{n \geq 0}^{\wedge} A(S_n)$$

(read ‘Beta’) is the free  $\beta$ -ring on one generator. It makes an early brief appearance in [Knu73], and plays a central role in [Rym77], [Och88], [MW84], [Val93], [Gui06], and [MS].

Let  $H \subseteq S_n$  be a subgroup. Then evaluation at  $H$  is an element of  $SCF(S_n)^*$ . Precomposing with  $\eta^{SCF}$ , this becomes the element of  $A(S_n)^*$ . Similarly,  $H$  defines the maps

$$\begin{aligned} SCF(S_n \times \underline{G}) &\longrightarrow SCF(\underline{G}) \\ \chi &\longmapsto \chi([H] \times -) \\ &\text{and} \\ A(S_n \times \underline{G}) &\longrightarrow A(\underline{G}) \\ Y &\longmapsto Y^H. \end{aligned} \tag{9}$$

#### 3.3.2 Linear and projective representations

If  $M = R$  is the representation ring functor then the Schur-Weyl theorem gives the Hopf algebra isomorphism

$$\bigoplus_{n \geq 0} R(S_n) \cong \lim_{n \geq 0} \mathbb{Z}[x_1, \dots, x_m]^{S_m}$$



identifying the representation ring of the symmetric groups with the Hopf ring of symmetric functions in infinitely many variables. On the left-hand side, the element  $f_n \in R(S_n)$  is the alternating representation, while  $c_k$  is the element satisfying

$$\langle \varrho, c_k \rangle_{S_k} = \chi_\varrho((1, \dots, k)).$$

On the right-hand side,  $f_n$  corresponds to the  $n$ th elementary symmetric function, while  $c_k$  becomes the  $k$ th power sum function.

We note that, differing from the standard convention, the inner product on  $R(\underline{G})$  is

$$\langle \varrho_1, \varrho_2 \rangle = (\varrho_1 \otimes \varrho_2)^G.$$

Let  $\alpha_n$  be the cocycle classifying Schur's spin extension

$$1 \longrightarrow \{\pm 1\} \longrightarrow \tilde{S}_n \longrightarrow S_n \longrightarrow 1$$

For  $n > 3$  Schur showed that  $[\alpha_n]$  is the only non-trivial element of  $H^2(S_n; U(1))$ . Following Józefiak, we write  $R^-(S_n)$  for the Grothendieck group of projective super-representations of  $S_n$  with cocycle  $\alpha_n$  and  $\Gamma = \mathbb{Z}[p_1(X), p_3(X), \dots]$  for the graded ring of polynomials in odd power sums of the variables  $x_1, x_2, \dots$  with  $\deg(p_j(X)) = j$ . Following Schur [Sch11], Józefiak constructs an isomorphism of graded  $\mathbb{Q}[\sqrt{2}]$ -algebras

$$\bigoplus_{n \geq 0} R^-(S_n)_{\mathbb{Q}[\sqrt{2}]} \xrightarrow{\cong} \Gamma \otimes \mathbb{Q}[\sqrt{2}].$$

### 3.3.3 Morava-Lubin-Tate theories

The Hopf ring

$$S_{E_n} = \bigoplus_{n \geq 0}^{\wedge} E_n(BS_n)$$

is the subject of [Str98] and [ST97].

### 3.3.4 n-Class functions

Recall that  $n$ -class functions are functions on

$$[\mathcal{T}^n \underline{G}] \cong \text{Prin}_{\underline{G}}(T^n) / \cong,$$

and that  $\text{Prin}_{S_k}(T^n)$  is equivalent to the category of  $k$ -fold covers of  $T^n$ . First, we consider the case  $\underline{G} = \text{pt.}$  The infinite groupoid

$$\coprod_{k \geq 0} \mathcal{T}^n S_k \cong \text{Cov}^{\text{fin}}(T^n)$$

is a monoid in  $\mathcal{Gpd}$ , with the concatenation product on the left-hand side corresponding to disjoint union of covers on the right. So, indecomposable covers are the connected covers. Indecomposable  $n$ -tuples of permutations are described as follows: let  $A$  be an abelian group together with a surjective group homomorphism

$$\alpha: \mathbb{Z}^n \twoheadrightarrow A.$$

Any numbering of the elements of  $A$  yields a map

$$A \twoheadrightarrow S_{|A|},$$

which we compose with  $\alpha$  to obtain an isomorphism class

$$[\alpha] \in [\mathcal{T}^n S_{|A|}],$$

independent of our choice of numbering. Elements of this form are exactly the indecomposable objects on the left. It follows that we have an isomorphism of graded rings

$$S_{\text{nClass}_R} \cong R[\chi'_\alpha]_{\alpha \in \mathfrak{A}'}$$

where  $\chi'_\alpha$  is the characteristic function of  $[\alpha]$ , with  $\deg(\chi'_\alpha) = |A|$ , while

$$\mathfrak{A}' = \{\alpha: \mathbb{Z}^n \twoheadrightarrow A\} / \cong$$

is the set of epimorphisms out of  $\mathbb{Z}^n$  up to isomorphisms under  $\mathbb{Z}^n$ . Similarly,

$$(S_{\text{nClass}_R})^{\text{Aut}(\mathbb{Z}^n)} \cong R[\chi_\alpha]_{\alpha \in \mathfrak{A}},$$

where function  $\chi_\alpha$  is the characteristic function of the  $\text{Aut}(\mathbb{Z}^n)$ -orbit of  $[\alpha]$ , and

$$\mathfrak{A} = \mathfrak{A}' / \text{Aut}(\mathbb{Z}^n).$$

Let now  $\underline{G}$  be arbitrary.

**Definition 3.5.** Let  $X$  be a topological space. The category  $\mathbf{G}\text{-Cov}_k(X)$  (respectively  $\mathbf{G}\text{-Cov}^{\text{fin}}(X)$ ) has as objects composites

$$\begin{array}{c} P \\ \downarrow \pi \\ Y \end{array} \xrightarrow{\phi} X,$$

where  $\pi$  is a principal  $G$ -bundle and  $\phi$  is an  $k$ -fold (respectively finite) cover. Morphisms in  $\mathbf{G}\text{-Cov}_k(X)$  are  $G$ -equivariant homeomorphisms  $f: P \rightarrow P'$  covering the identity on  $X$ .

For any such morphism  $f$ , the map  $(f/G): Y \rightarrow Y'$  is an isomorphism of  $k$ -fold covers of  $X$ .

**Lemma 3.6.** (a) *There is an equivalence of categories*

$$\text{Prin}_{S_k \int G}(X) \simeq \mathbf{G}\text{-Cov}_k(X).$$

(b) *Under these identifications, taking disjoint union of covers of degrees  $k$  and  $m$  corresponds to induction along the inclusion*

$$S_k \times S_m \hookrightarrow S_{k+m}.$$

(c) *Principal  $S_k \int S_m \int G$ -bundles are identified with composites*

$$P \xrightarrow{\pi} Z \xrightarrow{\chi} Y \xrightarrow{\phi} X,$$

where  $\pi$  is a principal  $G$ -bundle,  $\chi$  is a cover of degree  $m$  and  $\phi$  is a cover of degree  $k$ . Induction along the inclusion

$$\nabla: S_k \int S_m \int G \longrightarrow S_{km} \int G$$

corresponds to the functor that composes  $\chi$  and  $\phi$  to obtain an  $km$ -fold cover.

**PROOF OF LEMMA 3.6:** Fix  $X$ , and assume we are given an  $n$ -fold cover  $\phi: Y \rightarrow X$  and a principal  $G$ -bundle  $\pi: P \rightarrow Y$  over the total space of  $\phi$ . We let  $Q$  be the bundle whose fiber over  $x$  consists of all possible numberings of the fiber of  $Y$  over  $x$ . Then  $Q$  is a principal  $S_n$ -bundle over  $X$  whose associated  $n$ -fold cover is canonically isomorphic to  $Y$ . Similarly, we define

a bundle  $R$  over  $Q$  whose fiber over  $(y_1, \dots, y_n)$  consists of ordered  $n$ -tuples  $(p_1, \dots, p_n)$  of points in  $P$  satisfying  $\pi(p_i) = y_i$ . The action of  $G$  on  $P$  makes  $R$  into a principal  $G^n$ -bundle over  $Q$ , and the action of  $S_n$  on the fibers of  $R$  makes  $R$  into a principal  $S_n \wr G$ -bundle over  $X$ .

Conversely, assume that  $\psi: R \rightarrow X$  is a principal  $S_n \wr G$ -bundle. Let  $Q := G^n \backslash R$ , and consider the commuting diagram

$$\begin{array}{ccc} R & & \\ \xi \downarrow & \searrow \psi & \\ Q & \xrightarrow{\eta} & X, \end{array}$$

where  $\xi$  and  $\eta$  are the quotient maps. Then  $\xi$  naturally has the structure of a principal  $G^n$ -bundle, while  $\eta$  inherits that of a principal  $S_n$ -bundle.

We view  $S_{n-1}$  as a subgroup of  $S_n$  by identifying the elements of  $S_{n-1}$  with the permutations of  $\{1, \dots, n\}$  that fix  $n$ . Let

$$P := (S_{n-1} \wr G) \backslash R$$

and

$$Y := S_{n-1} \backslash Q,$$

and let  $\pi$  and  $\phi$  be the maps induced by  $\xi$  and  $\eta$ . Then  $\phi: Y \rightarrow X$  is the  $n$ -fold cover associated to  $\eta$ . We need to show that  $\pi: P \rightarrow Y$  is a principal  $G$ -bundle. If  $p \in P$  is the equivalence class of  $r \in R$ , we let  $gp$  be the equivalence class of

$$(\text{id}; 1, \dots, 1, g) \cdot r.$$

This action of  $G$  on  $P$  is well-defined, because the elements of  $\{1\}^{n-1} \times G$  commute with those of  $S_{n-1} \wr G$  in  $S_n \wr G$ . To identify the fibers of  $P$  over  $Y$  and over  $X$ , we note that we have a bijective map

$$\begin{array}{ccc} (S_{n-1} \wr G) \backslash (S_n \wr G) & \longleftrightarrow & (S_{n-1} \backslash S_n) \times G \\ \overline{(\sigma; g_1, \dots, g_n)} & \longmapsto & (\bar{\sigma}, g_n). \end{array}$$

Here  $\bar{x}$  denotes the coset of  $x$ . We conclude that  $\pi$  is indeed a principal  $G$ -bundle. It is straight-forward that the composites of the two functors

$$R \mapsto (P, Y) \quad \text{and} \quad (P, Y) \mapsto R$$

are naturally isomorphic to the respective identity functors.  $\square$

**Corollary 3.7.** *We have an equivalence of groupoids*

$$\coprod_{k \geq 0} \mathcal{I}^n(S_k \int \underline{G}) \simeq \underline{G}\text{-Cov}^{\text{fin}}.$$

The graded algebra  $S_{\text{n-Class}}(\underline{G})$  is the algebra of invariant functions on  $\underline{G}\text{-Cov}^{\text{fin}}$ . The element  $c_k$  is the function

$$c_k(Y \xrightarrow{\phi} T^n) = \begin{cases} 1 & \text{if } Y \text{ is connected} \\ 0 & \text{else.} \end{cases}$$

The element  $f_k$  is the function

$$f_k(Y \xrightarrow{\phi} T^n) = \prod_{Y = \coprod Y_i} (-1)^{k_i - 1},$$

where the product is over the connected components of  $Y$  and  $\phi|_{Y_i}$  is a  $k_i$ -fold cover of  $T^n$ .

### 3.3.5 Product decompositions

**Definition 3.8.** Let  $\mathcal{I}(\underline{G})[\xi^{\frac{1}{k}}]$  be the groupoid obtained from  $\mathcal{I}(\underline{G})$  by adding the additional arrows

$$(x, g) \xrightarrow{g^{\frac{1}{k}}} (x, g),$$

subject to the relations

$$(g^{\frac{1}{k}})^k = g \quad \text{and} \quad hg^{\frac{1}{k}} = (h^{-1}gh)^{\frac{1}{k}}h.$$

Then we have the equivalence of groupoids

$$\coprod_{n \geq 0} \mathcal{I}(S_n \int \underline{G}) \xleftarrow{\sim} \prod_{k \geq 1} \left( \coprod_{m \geq 0} S_m \int \left( \mathcal{I}(\underline{G})[\xi^{\frac{1}{k}}] \right) \right) \quad (10)$$

discussed, for instance in [Gan] (following [DMVV97]). As an immediate consequence of (10), we obtain the well known identity

$$\sum_{n \geq 0} \dim R(S_n)_{\mathbb{C}} t^n = \prod_{k \geq 1} \frac{1}{1 - t^k}.$$

One step higher, we obtain a bivariantly natural isomorphism

$$S_{2Class}(\underline{G}) \cong \bigotimes_{k \geq 1} S_{1Class} \left( \mathcal{I}(\underline{G})[\xi^{\frac{1}{k}}] \right)$$

and, in particular,

$$S_{2Class}(\text{pt}) \cong \bigotimes_{k \geq 1} S_{1Class}(\mathbb{Z}/k\mathbb{Z}).$$

A similar formula holds for Tate  $K$ -theory [Gan], but the product decomposition (10) is not compatible with the  $SL_2\mathbb{Z}$ -action.

**Question:** Let

$$\alpha_n = p_1(\varrho_n)$$

be the first Pontrjagin class of the permutation representation. It would be interesting to know whether there is a product decomposition for

$$S_{K_{Tate}}^- = \bigoplus_{n \geq 0}^{\wedge} K_{Tate}^{\alpha_n}(S_n)$$

in terms of the  $S_R^{\pm}(\mathbb{Z}/k\mathbb{Z})$  of Section 3.3.2.

### 3.3.6 $G$ -modular functions

In the case of  $G$ -modular functions, we have the graded rings

$$S_{Mod} = \bigoplus_{n \geq 0}^{\wedge} \Gamma(\mathcal{M}_{S_n}, \mathcal{O}_{\mathcal{M}_{S_n}})$$

and

$$S_{Mod}^- = \bigoplus_{n \geq 0}^{\wedge} \Gamma(\mathcal{M}_{S_n}, \mathcal{L}^{p_1(\varrho_n)}).$$

Recall that  $\mathcal{M}_{S_n}$  is the moduli space parametrizing principal  $S_n$ -bundles over complex elliptic curves or, equivalently,  $n$ -fold covers of complex elliptic curves.

**Example 3.9.** Let  $\Gamma \subseteq \langle \tau, 1 \rangle$  be the sublattice generated by  $d$  and  $\tau + b$  with  $0 \leq b < d$ . Then the degree  $d$ -cover

$$\phi: \mathbb{C}/\Gamma \longrightarrow \mathbb{C}/\langle \tau, 1 \rangle$$

has  $S_d$ -monodromy  $\varsigma_d$  along the circle  $[1, 0]$  and  $\varsigma^b$  along  $[0, \tau]$ . The source of  $\phi$  is isomorphic to  $\mathbb{C}/\langle \tau', 1 \rangle$  with  $\tau' = (\tau + b)/d$ .

**Example 3.10.** The degree  $a$  isogeny

$$\mathbb{C}/\langle a\tau, 1 \rangle \longrightarrow \mathbb{C}/\langle \tau, 1 \rangle$$

has  $S_a$ -monodromy 1 along  $[1, 0]$  and  $\varsigma_a$  along  $[0, \tau]$ .

Write

$$\mathcal{M}_{S_n \wr G} \cong \tilde{\mathcal{H}}_{G,n} \sqcup \mathcal{M}_{G,n}^{dec},$$

where  $\tilde{\mathcal{H}}_{G,n}$  parametrizes the covers with connected total space and  $\coprod \mathcal{M}_{G,n}^{dec}$  are the decomposable covers. Then the function  $c_n$  is

$$c_n|_{\tilde{\mathcal{H}}_n} \equiv 1 \quad \text{and} \quad c_n|_{\mathcal{M}_n^{dec}} \equiv 0.$$

It is believed that  $S_{Mod}^-$  has more interesting elements than the untwisted  $S_{Mod}$ . For instance, it seems reasonable to expect transfers of the generalized Moonshine functions to turn up as elements of  $S_{Mod}^-$ .

## 4 Global power functors

### 4.1 Definition

Let  $A$  be a commutative ring with 1, and let  $M$  be a global Green functor with values in  $A$ -mod.

**Definition 4.1.** A (total) power operation on  $M$  is a bivariantly natural, non-additive, transformation

$$P: M \Longrightarrow S_M$$

satisfying

**Exponentiality:** The map

$$P_\emptyset: M(\emptyset) \longrightarrow M(\text{pt})$$

maps 0 to  $1 \in A$ , and

$$P_{\underline{G} \sqcup \underline{H}}: M(\underline{G}) \oplus M(\underline{H}) \longrightarrow S_M(\underline{G} \sqcup \underline{H})$$

maps  $(a, b)$  to  $\kappa(P_{\underline{G}}(a) \otimes P_{\underline{H}}(b))$ .

**Comodule axiom:** We have

$$P \circ P = \nabla \circ P: M \implies S_{S_M}$$

and

$$\nu \circ P = \text{id}_M.$$

A *global power functor* is a global Green functor  $M$  together with a total power operation  $P$ .

#### 4.1.1 Consequences

Writing

$$P = \sum_{n=0}^{\infty} P_n t^n,$$

we have

**Cartan formula:**  $P(0) = 1_0$  and  $P(a + b) = P(a) \times P(b)$  (cross product).

**Low degrees:**  $P_1(x) = x$  and  $P_0(x) = 1$ .

**External product:**

$$\kappa(P_j(x) \otimes P_k(y)) = \text{res}_{S_j \times S_k}^{S_{j+k}} (P_{j+k})(x).$$

**Composition:**

$$P_j(P_k(x)) = \text{res}_{S_j \wr S_k}^{S_{jk}} (P_{jk})(x)$$



**Multiplicativity:**

$$P_j(1) = 1 \quad \text{and} \quad P_j(ab) = P_j(a)P_j(b)$$

(juxtaposition).

**External products:** Let

$$\delta: S_j \int (\underline{G} \times \underline{H}) \rightarrow (S_j \int \underline{G}) \times (S_j \int \underline{H})$$

be the map induced by the diagonal inclusion of  $S_j$  in  $S_j \times S_j$ . Then we have

$$P_j(\kappa(x \otimes y)) = \text{res}_\delta \kappa(P_j(x) \otimes P_j(y)).$$

## 4.2 $\tau$ -rings and $\lambda$ -rings

Let  $M$  be a global power functor whose Künneth maps are isomorphisms. Following Hoffman [Hof79], we define the total  $\tau$ -operation

$$\tau: M\underline{G} \xrightarrow{P} S_M(\underline{G}) \xrightarrow{\delta^*} S_M \otimes M\underline{G},$$

and the internal operations  $\tau^a$ , with  $a \in M(S_n)$ ,

$$\tau^a: M\underline{G} \xrightarrow{\tau_n} M(S_n) \otimes M\underline{G} \xrightarrow{\langle -, a \rangle \otimes \text{id}} M\underline{G},$$

where

$$\tau = \sum_{n \geq 0} \tau_n t^n.$$

**Remark 4.2** (Variations). If each  $M(S_n)$  is self-dual via  $\langle -, - \rangle$  then the internal operations  $\tau^a$  with  $a$  in (a basis of)  $M(S_n)$  determine  $\tau_n$ . If there is no Künneth isomorphism then  $\tau_n$  takes values in  $M(S_n \times \underline{G})$ . In this case, one can still define the internal operations, using the push-forward along the map  $pr_2: S_n \times \underline{G} \rightarrow \underline{G}$ . One can also replace  $\langle -, a \rangle$  by any  $M(pt)$ -module map  $\phi: M(S_n) \rightarrow N$  to obtain an operation  $\tau^\phi$  with values in  $N \otimes M(\underline{G})$ . This variation is useful when the inner product is not defined.

Hoffman proves that these operations satisfy the  $\tau$ -ring axioms:

**Cartan formula:**  $\tau(0) = 1_0 \otimes 1$  and

$$\tau(x + y) = \tau(x) \times \tau(y),$$

where  $\times$  is the cross product tensored with multiplication in  $M(\underline{G})$ . In particular, the map

$$\begin{aligned} M(\underline{H}) \oplus M(\underline{G}) &\longrightarrow S_M(\underline{H}) \hat{\otimes} S_M \otimes M(\underline{G}) \\ (x, y) &\longmapsto P(x) \otimes \tau(y) \end{aligned}$$

is bilinear from  $+$  to  $\times$ , yielding a unique exponential extension

$$M(\underline{H}) \otimes M(\underline{G}) \longrightarrow S_M(\underline{H}) \hat{\otimes} S_M \otimes M(\underline{G}),$$

which we will denote  $P \otimes \tau$ .

**Low degrees:**  $\tau_0(x) = 1_0 \otimes 1$  and  $\tau_1(x) = 1_1 \otimes x$ .

**External products:**

$$m: \tau_j(x) \otimes \tau_k(x) \longmapsto \text{res}_{S_j \times S_k}^{S_{j+k}} \tau_{j+k}(x),$$

where  $m$  is  $id_{M(S_j \times S_k)}$  tensored with multiplication in  $M(\underline{G})$ .

**Composition:**

$$(P \otimes \tau)_j \circ \tau_k(x) \longmapsto \text{res}_{S_j \int S_k}^{S_{j+k}} \tau_{j+k}(x)$$

under the map

$$(id \cdot \Pi^*) \otimes id : M(S_j \int S_k) \otimes M(S_j) \otimes M(\underline{G}) \longrightarrow M(S_j \int S_k) \otimes M(\underline{G}),$$

where  $\Pi: S_j \int S_k \longrightarrow S_j$  is the projection.

**Multiplicativity:**  $\tau_j(1) = 1_j \otimes 1$  and  $\tau_j(xy) = \tau_j(x) \cdot \tau_j(y)$  (juxtaposition).

Still following Hoffman, we find that consequences of these axioms include

**Universal maps:** For fixed  $x \in M(\underline{G})$ , we have a ring map

$$\begin{aligned} U_x: S_M &\longrightarrow M(\underline{G}) \\ a &\longmapsto \tau^a(x) \end{aligned}$$

sending  $1 \in M(S_1)$  to  $x$ . Here the product on the source is the cross product.

**Outer plethysm:** Composition of the internal operations is given by the formula

$$\tau^b \circ \tau^a = \tau^{b \vee a}$$

with

$$\begin{aligned} \vee : R(S_j) \otimes R(S_k) &\longrightarrow R(S_{jk}) \\ b \vee a &= \nabla_* (P_j(a) \cdot \Pi^*(b)) \end{aligned}$$

the *outer plethysm* [Hof79, Prop.4.5].

**Definition 4.3.** A  $\tau$ -ring with respect to the global power functor  $M$  is a commutative and unitary  $M(\text{pt})$ -algebra  $E$  together with an operation

$$\tau : E \longrightarrow S_M \otimes E$$

satisfying the  $\tau$ -ring axioms. If  $M$  is clear from the context ‘ $\tau$ -ring’ will always be meant with respect to  $M$ .

The discussion above implies that the ring  $(S_M, \times, 1_0)$  with the operations

$$\begin{aligned} \tau_j(b) &= (\nabla, \Pi)_* P_j(b) \quad \text{and} \\ \tau^a(b) &= a \vee b \end{aligned}$$

is the free  $\tau$ -ring on the generator  $1_1$ . Here  $\nabla$  is the inclusion of  $S_j \int S_k$  in  $S_{jk}$  and  $\Pi$  is the projection to  $S_j$ , where the degree of  $b$  is  $k$ .

On any  $\tau$ -ring  $E$ , we have the operations

$$\begin{aligned} \text{sym}^n &= \tau^{1_n} && \text{symmetric powers,} \\ \lambda^n &= \tau^{f_n} && \lambda\text{-operations, and} \\ \psi^n &= \tau^{c_n} && \text{Adams operations.} \end{aligned}$$

Here  $1_n$ ,  $f_n$  and  $c_n$  are as in Section 3.2.1. The  $\lambda^n$  make  $E$  a (not necessarily special)  $\lambda$ -ring. In particular, the Adams operations are additive. We also have the operations

$$\psi^H = \tau^{\eta(X_H)},$$

where  $H$  is a finite group and  $\eta(X_H)$  is as in Example 2.24. Write  $P_H(x)$  for the restriction of  $P_n(x)$  along the inclusion

$$H \times \underline{G} \subseteq S_n \int \underline{G},$$

and let  $\epsilon_H: H \times \underline{G} \rightarrow \underline{G}$  be the projection to the second factor. Then, by Frobenius reciprocity,

$$\psi^H(x) = (\epsilon_H)_! (P_H(x)).$$

## 4.3 Examples

### 4.3.1 (Special) $\lambda$ -rings

The representation ring functor  $\underline{G} \mapsto R(\underline{G})$  is a global power functor with

$$P_n(V) = V^{\otimes n}$$

[Ati66]. Here  $\text{sym}^n$  and  $\lambda^n$  are the familiar symmetric and exterior powers, and  $\psi^n$  are the Adams operations. Hoffman shows that  $R$ -theoretic  $\tau$ -rings are the same thing as special  $\lambda$ -rings.<sup>3</sup> The universal map

$$\begin{aligned} U_{\text{id}} : S_R &\longrightarrow R(U) \quad \text{or} \\ U_X : S_R &\longrightarrow \lim_m \mathbb{Z}[x_1, \dots, x_m]^{S_m} \end{aligned}$$

$X = x_1 + x_2 + x_3 + \dots$  is the Schur-Weyl map.<sup>4</sup>

### 4.3.2 Super $\lambda$ -rings, Sergeev-Yamaguchi duality

**Definition 4.4.** A  $\mathbb{Z}/2\mathbb{Z}$ -graded groupoid is a groupoid  $\underline{G}$  together with a map of groupoids

$$\underline{G} \longrightarrow \mathbb{Z}/2\mathbb{Z}.$$

---

<sup>3</sup>This term is somewhat old-fashioned. Nowadays, ‘special  $\lambda$ -rings’ are often simply referred to as ‘ $\lambda$ -rings’. Since we are interested also in non-special examples, we stick with the old-fashioned terminology.

<sup>4</sup>Note that there is a mismatch here with our conventions for the inner products: to get the correct map, one needs  $\langle \varrho, \vartheta \rangle = \text{Hom}_G(\varrho, \vartheta)$ . Switching conventions amounts to passing from  $V_\lambda$  to  $V_\lambda^*$ .

For such  $\underline{G}$ , we let  $R(\underline{G})$  be the Grothendieck group of super-representations of  $\underline{G}$  with the additional relation

$$[V] + [\Pi V] = 0,$$

where  $\Pi$  is the shift functor.<sup>5</sup> We recall from [GK11] that  $R(\underline{G})$  itself may be viewed the degree zero part of a superring: let  $\mathcal{C}_1 = \mathbb{C}[\xi_1]/\xi_1^2$  be the Clifford superalgebra  $|\xi_1| = 1$ , and write  $\otimes_s$  for the super tensorproduct. Then

$$R(\underline{G})_1 = R(\mathbb{C}[\underline{G}] \otimes_s \mathcal{C}_1)$$

The product map

$$\otimes: R(\underline{G})_\bullet \otimes_s R(\underline{H})_\bullet \xrightarrow{\cong} R(\underline{G} \times \underline{H})_\bullet$$

is defined as follows (compare [Kle05, 12.21]): If  $V$  and  $W$  are irreducible and  $\deg[V] \cdot \deg[W] = 0$ , then

$$V \otimes W = V \otimes_s W.$$

If  $\deg(V) = \deg(W) = 1$ , then  $V \otimes W$  is the inverse image of  $V \otimes_s W$  under the periodicity isomorphism

$$R(\underline{G}) \xrightarrow{\cong} R(\mathbb{C}[\underline{G}] \otimes_s \mathcal{C}_2),$$

$\mathcal{C}_2 = \mathcal{C}_1^{\otimes_s 2}$ . In other words,

$$V \otimes W = (V \otimes_s W)^+$$

is the  $+1$ -eigenspace of  $i\xi_1\xi_2$  inside  $V \otimes_s W$ .

The product  $\otimes$  makes  $R(-)_\bullet$  a globally defined Green functor on the category of  $\mathbb{Z}/2\mathbb{Z}$ -graded groupoids. In particular, each  $R(\underline{G})_\bullet$  is a superring.

We have power operations on  $R_\bullet$ , defined as follows:

$$\begin{aligned} P_k^+ : R(\underline{G})_0 &\longrightarrow R(S_k \int \underline{G})_0 \\ P_k^- : R(\underline{G})_1 &\longrightarrow R^-(S_k \int \underline{G})_{\overline{k}} \\ V &\longmapsto V^{\otimes k}, \end{aligned}$$

---

<sup>5</sup>This differs from the convention in [Józ89] and [Kle05, 12.18 ff.] by a sign.

where  $\bar{k}$  is the parity of  $k$ , and  $S_k$  acts by permuting the factors. In the case of  $P^-$  this makes  $V^{\otimes k}$  a spin representation of  $\widetilde{S}_k$  (in a somewhat non-canonical manner, see [Yam99]).

$R(\underline{G})$  is the prototype of a *super  $\lambda$ -ring*: a super ring  $A_\bullet$  together with operations

$$\begin{aligned}\tau^+ : A_0 &\longrightarrow (A_\bullet \otimes_s S_{R_\bullet}^+)_0 \\ \tau^- : A_1 &\longrightarrow A_\bullet \otimes_s S_{R_\bullet}^-\end{aligned}$$

with  $\tau_k^-$  taking values in degree  $k$  and such that  $\tau = (\tau^+, \tau^-)$  satisfies the  $\tau$ -ring axioms.

**Example 4.5** (Sergeev-Yamaguchi duality). Let  $A = R(\mathfrak{q}(n))_\bullet$  be the representation ring of the queer Lie superalgebra  $\mathfrak{q}(n)$ , and let  $V$  be the defining representation of  $\mathfrak{q}(n)$ . The main result of [Yam99] may be summarized as

$$[V^{\otimes k}] = \sum_{\substack{\lambda \vdash k \\ \text{strict}}} [V_\lambda] \circledast [W_\lambda],$$

where  $V_\lambda$  runs through the irreducible spin superrepresentations of  $\widetilde{S}_n$  and the  $W_\lambda$  are pairwise distinct irreducible superrepresentations of  $\mathfrak{q}(n)$ . It follows that the universal map

$$U_V^- : S_{R_\bullet}^- \longrightarrow R(\mathfrak{q}(n))_\bullet$$

maps  $a = [V_\lambda]$  to

$$(\tau^-)^{[V_\lambda]}(V) = [W_\lambda].$$

**Remark 4.6.** We chose here to view  $R(\underline{G})_\bullet$  as a  $\mathbb{Z}/2\mathbb{Z}$ -graded theory. This made it a natural thing to consider ‘*Yamaguchi power operations*’  $V \mapsto V^{\otimes k}$  as above. Yamaguchi explains in [Yam99, Sec.3] how to translate between his picture and that of Sergeev: in the language of [GK11, Sec (1.7)], the ‘*Sergeev power operations*’ act on the  $\mathbb{N}$ -graded theory

$$\begin{aligned}R(\underline{G})_m &\longrightarrow R^-(S_k \times \underline{G})_{mk} \\ V &\longmapsto V^{\otimes_s k}\end{aligned}$$

and correspond to the Yamaguchi power operations under the periodicity isomorphism.

### 4.3.3 $n$ -special $\lambda$ -rings

As above, we view

$$\chi \in \text{n-Class}(\underline{G}, R)^{SL_n(\mathbb{Z})}$$

as  $SL_n(\mathbb{Z})$ -invariant global function on  $\text{Prin}_{\underline{G}}(T^n)$ .

**Definition 4.7.** The function  $P_k(\chi)$  on  $\underline{G}\text{-Cov}_k(T^n)$  is defined by

$$(P_k \chi) \left( P \xrightarrow{\pi} Y \xrightarrow{\phi} T^n \right) = \prod_{Y = \coprod Y_i} \chi(P|_{Y_i} \xrightarrow{\pi_i} Y_i),$$

where the product runs over all connected components of  $Y$ .

To make sense of the factors on the right-hand side, pick, for each  $i$ , a basis of the torus  $Y_i$ . Since  $\chi$  is invariant under the  $\text{Aut}(\mathbb{Z}^n)$ -actions, its value at  $\pi_i$  is independent of this identification  $Y_i \cong T^n$ . We will refer to the corresponding  $\tau$ -rings as  $n$ -special  $\lambda$ -rings. The internal power operations on an  $n$ -special  $\lambda$ -ring are generated by the ‘*elementary*’ power operations  $\tau^{\underline{\sigma}(\alpha)}$ , where the  $n$ -tuple

$$\underline{\sigma}(\alpha) : \mathbb{Z}^n \xrightarrow{\alpha} \twoheadrightarrow A \rightarrowtail S_{|A|}$$

is viewed as the linear form on  $\text{n-Class}(S_{|A|}, R)$  sending  $\chi$  to  $\chi(\underline{\sigma}(\alpha))$ . These are calculated as follows: Let

$$\varphi : \ker(\alpha) \hookrightarrow \mathbb{Z}^n$$

be the inclusion, and choose any oriented basis of the lattice  $\ker(\alpha)$ . For  $\beta : \mathbb{Z}^n \rightarrow \underline{G}$ , we then have

$$(\tau^{\underline{\sigma}(\alpha)}(\chi))(\beta) = \chi(\beta \circ \phi^t).$$

If  $R$  contains  $\mathbb{Q}$ , so inner products exist, then we have symmetric and exterior powers, and the discussion in [Gan06, Sec.9] goes through to prove that the Adams operators of the theory are

$$\psi^k = \sum_{|A|=k} \tau^{\underline{\sigma}(\alpha)}.$$

#### 4.3.4 $\beta$ -rings and subgroup characters

The Burnside ring functor  $A(\underline{G})$  is a global power functor via

$$P_k: [X] \longmapsto [X^k].$$

The Künneth maps for  $A$  are not isomorphisms, but one can still make sense of a ring with internal  $\tau$ -operations with respect to  $A$ . These internal operations are generated by the  $\beta$ -operations

$$\beta_H = \tau^{[S_k/H]},$$

where  $H$  ranges over the conjugacy classes of (transitive) subgroups of  $S_k$ . This point of view is taken in [Rym77] and was developed further in [Och88], [MW84], [Val93], [Gui06], [BR09], and [MS]. The resulting formalism is that of  $\beta$ -rings. These can be rather complicated objects, Guillot shows that examples include the stable homotopy groups of spheres and, more generally, the stable cohomotopy ring of a space.

Let us now turn to the related Mackey functor  $SCF$  of subgroup class functions, where the subgroup characters of permutation representations take their values. An idea going back to Knutson is to look for a Schur-Weyl type map out of  $\mathbf{B}$ , using  $SCF$ . For this, Knutson suggests a definition of Adams operations on  $SCF$  and conjectures that the subgroup character map  $\eta^{SCF}$  is a map of  $\lambda$ -rings. The above discussion suggests a definition of power operations (and hence Adams-operations) on  $SCF$ , which is different to the structure anticipated by Knutson. The transformation  $\eta^{SCF}$  preserves power operations. However, since the surjections axiom is not satisfied,  $\eta^{SCF}$  does not commute with transfers along the maps  $\epsilon_G: \underline{G} \rightarrow 1$ , so that care must be taken when trying to use it to analyze internal operations.

For simplicity, we let  $\underline{G}$  be a group  $G$ . Let  $H$  be a subgroup of  $S_n \wr G$ . Then we have an  $n$ -fold covering space  $Y$  of  $BH$ , together with a principal  $G$ -bundle  $\pi: P \rightarrow Y$ . The space  $Y$  is connected if and only if the image of the projection  $H \rightarrow S_n$  is transitive. In that case we have  $Y = EH/K$  for some subgroup  $K \subseteq H$  with index  $[H : K] = n$ , and  $P$  is classified by a map  $K \rightarrow G$ , well defined up to inner automorphism of  $G$ . The image of this map defines an element  $[K]_G$  of  $[S(G)]$ . If  $H$  is not transitive, then  $H = \prod H_i$  is the product of transitive subgroups  $H_i \subseteq S_{n_i}$ , and we get subgroups  $K_i \subseteq H_i$  of index  $n_i$ , each of which gives an element of  $[S(G)]$ .



**Definition 4.8.** Define power operations

$$\begin{aligned} P_n : SCF(G) &\longrightarrow SCF(S_n \wr G) \\ \chi &\longmapsto \left( P_n(\chi) : [H] \longmapsto \prod_i \chi([K_i]_G) \right). \end{aligned}$$

In the spirit of [Gui06], Knutson's ‘Schur-Weyl’ map should take values in the Burnside ring of the infinite general linear group of the field with one element. A possible candidate is the map

$$U : \mathbf{B} \longrightarrow \lim_{m \in \mathbb{N}} A(S_m).$$

Here  $[m]$  is the set with  $m$  elements and the defining  $S_m$ -action, and  $U_{[m]}$  is the map

$$\begin{aligned} U_{[m]} : \mathbf{B} &\longrightarrow A(S_m) \\ [S_k/H] &\longmapsto \tau^{\eta_H}([m]). \end{aligned}$$

Note that we have used  $\tau^{\eta_H}$  instead of  $\tau^{[S_k/H]}$ , where

$$\eta_H : A(S_k \times \underline{G}) \longrightarrow A(\underline{G})$$

picks out the subset where  $S_k$  acts with orbit type  $S_k/H$ . To study the map  $U$ , one needs an understanding of the simultaneous decomposition of

$$[m]^k = \text{Map}([k], [m])$$

into  $S_k$ - and  $S_m$ -orbit types. Alternatively, one could modify this construction to take values in

$$\lim_{m \in \mathbb{N}} SCF(S_m).$$

#### 4.3.5 $G$ -modular functions

Let  $F$  be a global function on  $\mathcal{M}_G$ . Then the global function  $P_n(f)$  on  $\mathcal{M}_{S_n \wr G}$  is defined as follows: Given an elliptic curve  $E$  and a principal  $S_n \wr G$ -bundle  $\psi : P \rightarrow E$ , we set

$$P_n(F)(\psi) = \prod_{Y = \coprod Y_i} F(\pi_i),$$

where the sum is over the connected components of  $Y = [\underline{G} \ltimes P]$ . Here we are taking advantage of the fact that the maps  $\phi_i: Y_i \rightarrow E$  are isogenies and that this identification is canonical after choosing basepoints of the  $Y_i$ , a choice which does not affect  $F(\pi_i)$ . The resulting internal power operations play a big role in [Gan09]. In particular, the Adams operations of the theory are the  $G$ -equivariant Hecke operators.

#### 4.3.6 Rezk's elliptic $\lambda$ -rings

We will work with  $q$  inverted. In [Gan], we defined the total power operation on  $K_{\text{Tate}}$  as follows

$$P_t^{\text{Tate}}(F)(q) = \prod_{k \geq 1}^{\times} P_{t^k}^{\text{Atiyah}}(\theta_k(F))(q),$$

where  $\theta_k(F)(q)$  is the power series over  $R\left(\mathcal{I}(\underline{G})[\xi^{\frac{1}{k}}]\right)$  with character

$$(\theta_k F)\left(g, hg^{\frac{b}{k}}; q\right) = F\left(g, h; q^{\frac{1}{k}} \zeta_k^b\right),$$

and  $P^{\text{Atiyah}}$  is defined by

$$P_n^{\text{Atiyah}}(V q^{\frac{m}{k}}) = V^{\otimes n} q^{\frac{mn}{k}}.$$

Again, the Adams operations of the theory are the ( $q$ -expansions of) equivariant Hecke operators. The total symmetric power of  $[V] \in R(\underline{G}) \subseteq K_{\text{Tate}}(\underline{G})$  is Witten's class

$$S_t^{\text{Tate}}(V) = \bigotimes_{k \geq 0} S_{t^k}^{\text{Atiyah}}(V)$$

(see [Gan]).

**Definition 4.9.** Let  $r_n$  be the regular representation of the centralizer of  $\varsigma_n$  and define the sequences  $\delta_n$  and  $\gamma_n$  by letting

$$\begin{aligned} \delta_n &= ([1], \varsigma_n) \\ \gamma_n &= ([\varsigma_n], r_n). \end{aligned}$$

Note that  $\delta_n \in K_{Tate}(S_n)$ , while  $\gamma_n$  does not satisfy the rotation condition. Then any  $K_{Tate}$  theoretic tau-ring has the internal operations

$$\nu^n = \tau^{\delta_n}$$

(denoted  $\psi^n$  in [Rezb]) and the operations

$$\mu^n = \tau^{\gamma_n}$$

mapping from  $A$  to  $A[q^{\frac{1}{n}}]$ .

These operations make our  $K_{Tate}$ -theoretic tau rings into *elliptic lambda rings* as defined by Rezk [Rezb]. In particular,  $K_{Tate}(\mathbb{Z}/N\mathbb{Z})$  is an elliptic lambda ring. This is the ring  $T[N]$  considered in [Rezb], with

$$\text{spec}(K_{Tate}(\mathbb{Z}/k\mathbb{Z})) \cong \text{Tate}(q)[k]$$

the scheme of  $k$ -torsion points of the Tate curve.

**Theorem 4.10.** *The  $q$ -expansion map of Section 2.4.9 is a map of global power functors.*

PROOF : Let  $F \in \text{Mod}(\underline{G})$ . It suffices to calculate the  $q$ -expansion of  $(P_n F)|_{\tilde{H}_n}$  in terms of the  $q$ -expansion of  $F$ . Let  $\phi$  be an isogeny onto  $\mathbb{C}/\langle \tau, 1 \rangle$ . If  $\phi$  is as in Example 3.9 then the discussion there implies that

$$(P_d(F))(\phi, (g, h); \tau) = F\left(g, h; \frac{\tau + b}{d}\right).$$

This is the same formula as for  $\theta_d$ . The isogeny  $\phi$  corresponds to the commuting pair  $(\varsigma_d, \varsigma_d^b)$  in  $S_d$ , so that  $(\phi, g, h)$  is identified with  $(g, hg^{\frac{b}{d}}) \in \mathcal{I}(\underline{G})[\xi^{\frac{1}{d}}]$ .

If  $\phi$  is as in Example 3.10, then

$$(P_a(F))(\phi, (g, h); \tau) = F(g, h; a\tau).$$

This is the same as  $P_a^{\text{Atiyah}}(F)_{(1, \varsigma_a)}$ . An arbitrary isogeny may be written as composite of two isogenies as above.  $\square$

**Definition 4.11** (Looijenga's symmetric theta functions). Let  $G$  be a complex reductive algebraic group with maximal torus  $T$  and Weyl group  $W_0$ . Let

$$\begin{aligned}\widehat{T} &= \operatorname{Hom}(T, \mathbb{C}^\times) \\ \check{T} &= \operatorname{Hom}(\mathbb{C}^\times, T)\end{aligned}$$

be the character and cocharacter lattices of  $T$ , and let  $I$  be the minimal positive definite  $W_0$ -invariant symmetric bilinear form on  $\check{T}$  satisfying  $(\forall v \in \check{T})$

$$I(v, v) \in 2\mathbb{Z}. \quad (11)$$

Write

$$I^\sharp: \check{T} \longrightarrow \widehat{T}$$

for the adjoint of  $I$ . Consider formal power series  $f(z, q)$  with exponents of  $z$  in  $\widehat{T}$  and exponents of  $q$  in  $\mathbb{Z}$ , and  $z^\lambda z^\mu = z^{\lambda+\mu}$  and  $q^m q^n = q^{m+n}$ , and let

$$\Theta_I = \left\{ f(z, q) \mid (\forall v \in \check{T}) \left( f(zq^v, q) = q^{-\frac{I(v,v)}{2}} z^{-I^\sharp(v)} f(z, q) \right) \right\}.$$

Looijenga's ring of symmetric theta functions is

$$\Theta^{W_0} = \sum_{k \geq 0} \Theta_{kI}^{W_0}$$

( $W_0$ -invariant elements).

An example is the basic theta function

$$\theta_I = \sum_{v \in \check{T}} q^{\frac{I(v,v)}{2}} z^{I^\sharp(v)}.$$

Elements of  $\Theta^{W_0}$  should be interpreted as the sections of the Looijenga line bundle on  $\check{T} \otimes \operatorname{Tate}(q)$  and hence as the coefficients of  $G$ -equivariant Tate  $K$ -theory. These have yet to be defined, but see [Reza]. The action of isogenies of the Tate curve on Looijenga theta functions was calculated in [And00, Part III]. These should be internal  $\tau$ -operations with respect to  $K_{\operatorname{Tate}}$ .

The ring  $\Theta^{W_0}$  is the elliptic cohomology analogue of the rings  $R(U(n))$ . The basic theta function  $\theta_I$  plays the role of the defining representation  $[\mathbb{C}^n] \in R(U(n))$ . Hence we expect a Schur-Weyl map

$$U_{\theta_I}: S_{K_{\operatorname{Tate}}} \longrightarrow \Theta^{W_0},$$

which should be accessible via Ando's calculations. It would be interesting to study this map in more detail.

### 4.3.7 Generalized cohomology theories

If the generalized cohomology theory  $E$  is equipped with  $H_\infty$ -structure, then  $E(\text{Borel}(\underline{G}))$  is a global power functor. This is true for many familiar cohomology theories, such as ordinary cohomology, Lubin-Tate-Morava  $E$ -theory, topological modular forms,  $K$ -theory, and cobordism. The standard reference for  $H_\infty$ -spectra is [BMMS86], for a discussion of power operations on the Morava-Lubin-Tate theories and the corresponding  $\lambda$ -ring operations, see [And95], [Bak98], [Rez08], [Rez12], and [Gan06].

### 4.3.8 Character maps

**Proposition 4.12.** *The character map*

$$\text{char}: R \implies 1\text{Class}(-, \mathbb{Z}[\mu_\infty])^{\text{Aut}(\mathbb{Z})}$$

*preserves power operations.*

PROOF : Let  $\varrho$  be a  $\underline{G}$ -representation, and recall that  $P_n(\varrho)$  is the  $S_n \int \underline{G}$ -representation  $\varrho^{\otimes n}$ . We need to show

$$\chi_{\varrho^{\otimes n}}(\sigma; g_1, \dots, g_n) = \prod_{(i_1, \dots, i_k)} \chi_{\varrho}(g_{i_k} \cdots g_{i_1}),$$

where the product is over all cycles of  $\sigma$ . Indeed, the action of  $(\sigma; g_1, \dots, g_n)$  on  $V^{\otimes n}$  can be written as the tensor product over the cycles of sigma of actions of  $(\varsigma_k; g_{i_1}, \dots, g_{i_k})$  on  $V^{\otimes k}$ , with

$$\varsigma_k = (1, \dots, k) \in S_k.$$

Further

$$(\varsigma_k; g_{i_1}, \dots, g_{i_k}) \sim (\varsigma_k; 1, \dots, 1, g),$$

where  $g = g_{i_k} \cdots g_{i_1}$ . Let  $B = (e_1, \dots, e_d)$  be a basis of  $V$ . Then  $(\varsigma_k; 1, \dots, g)$  sends the basis element  $e_{j_1} \otimes \dots \otimes e_{j_k}$  of  $V^{\otimes k}$  to

$$(ge_{j_k}) \otimes e_{j_1} \otimes \dots \otimes e_{j_{k-1}}.$$

This basis element can only make a non-trivial contribution to the trace on  $V^{\otimes k}$  if we have  $j_1 = \dots = j_k$ , and in this case the contribution to the trace

is the  $j_1^{th}$  diagonal entry of  $g$ . We conclude that the trace of  $(\varsigma_k; 1, \dots, 1, g)$  on  $V^{\otimes k}$  equals the trace of  $\text{Mat}_B(g)$  on  $V$ .  $\square$

It would be nice to have a more conceptual proof of the proposition. The Hopkins-Kuhn-Ravenel character maps

$$\text{char}: E_n(B\underline{G}) \longrightarrow \text{n-Class}_p(\underline{G}; L)^{\text{Aut}(\widehat{\mathbb{Z}}_p)^n}$$

form a natural transformation of Green functors. Note that  $\text{Aut}(\widehat{\mathbb{Z}}_p^n)$  also acts on  $L$ , so this is not a special case of the discussion in Section 4.3.3. Instead, the situation is closer to  $\text{Mod}(\underline{G})$ . The effect of  $\text{char}$  on power operations is described using isogenies of formal groups. This is the subject of [And95] and [AHS04].

## A Comparison to Webb's definition

In this section, we will compare our definition of global Mackey functor with that of [Web00].

**Proposition A.1.** *The axioms for global Mackey functors stated in Section 2 imply the Axioms for global Mackey functors formulated in [Web00, 8], where Webb's classes  $\mathcal{X}$  and  $\mathcal{Y}$  are chosen to contain all finite groups.<sup>6</sup>*

PROOF : The Mackey axiom (double coset formula) follows from Example 2.7. One checks that a diagram of groups

$$\begin{array}{ccc} G & \xrightarrow{\beta} & H \\ \uparrow & & \uparrow \\ \beta^{-1}(K) & \longrightarrow & K \end{array}$$

is equivalent to a fibred-product square in  $\mathcal{G}\text{pd}_{\text{fin}}$ . Consider now a diagram of surjective maps of groups

$$\begin{array}{ccc} G & \xrightarrow{\gamma} & H / \ker \alpha \ker \beta \\ \uparrow \beta & & \uparrow \delta \\ H & \xrightarrow{\alpha} & K. \end{array}$$

---

<sup>6</sup>Webb attributes this formulation to Bouc [Bou96].

In general, this diagram is not a fibred-product square. However, the fibred product  $\underline{F} := G \times_{H/\ker \alpha \ker \beta} K$  is equivalent to a finite group (namely the pullback of the diagram in the category of finite groups), and under this equivalence, the map  $H \rightarrow \underline{F}$  becomes a surjective map of groups. Now our fibred-product axiom together with the surjection axiom imply the desired push-pull property for this square.  $\square$

**Proposition A.2.** *The axioms for global Mackey functor formulated in [Web00] imply ours.*

PROOF : Let  $M$  be a global Mackey functor in the sense of [Web00, 8]. Set  $M(\emptyset) = 0$ . Recall that every groupoid is equivalent to a disjoint union of finite groups. Let  $\underline{G}$  be a connected groupoid, and let  $x, y \in G_0$ . Let  $g$  be an arrow from  $x$  to  $y$  in  $G_1$ . Then conjugation by  $g$  defines an isomorphism from  $c^g: \text{Stab}(x) \rightarrow \text{Stab}(y)$ . For a different choice of arrow,  $g'$ , the two isomorphisms  $c^g$  and  $c^{g'}$  will differ from each other by an inner automorphism of  $\text{Stab}(y)$ . Hence the (inverse) isomorphisms  $(c^g)_*$  and  $(c^g)^*$  are independent of the choice of  $g$ . We set

$$M(\underline{G}) := \lim_{x \in G_0} M(\text{Stab}(x)).$$

Then for each  $x \in G_0$ , there is an isomorphism  $\text{Stab}(x) \cong M(\underline{G})$ . This definition extends to maps between (connected) groupoids by restricting to stabilizers. By construction, naturally isomorphic maps will yield the same result, and equivalences of groupoids will yield isomorphisms. We extend  $M$  to non-connected groupoids using our coproduct axiom (and the consequence for projections). Because of the coproduct axiom and the Note that our surjection axiom follows from Webb's axiom (4) in the special case that  $G = K$ . It suffices to check the fibred-product axiom for maps of finite groups

$$H \xrightarrow{\alpha} G \xleftarrow{\beta} K.$$

Each map of groups can be factored into a surjection followed by an injection, and fibred products compose as they should, so that we are reduced to three cases:

**Case 1** Both maps are injective: this case follows from Webb's Axiom (5) (the Mackey axiom).

**Case 2** One of the maps is injective and the other is surjective: this case follows from Webb’s Axiom (3).

**Case 3** If  $\alpha$  and  $\beta$  both are surjective, then the fibred product  $H \times_G K$  is equivalent to a finite group, namely to

$$L := \{(h, k) \mid \alpha(h) = \beta(k)\}.$$

We have surjections  $\gamma: L \rightarrow K$  and  $\delta: L \rightarrow H$  and an isomorphism

$$\begin{aligned} L/(\ker \gamma)(\ker \delta) &\cong L/\{(h, k) \mid \alpha(h) = 1 = \beta(k)\} \\ &\cong G. \end{aligned}$$

Using Webb’s Axiom (4), we obtain the desired push-pull identity.

□

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